DROP IN AN OSCILLATING LIQUID

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UDC 532.529.5

At the present time there is a complete lack of studies devoted to the forced motion of a drop located in an oscillating liquid. However, this problem is of considerable interest. For example, it represents simulation of hydrodynamic processes occurring during the irradiation of drops of one liquid located in another liquid by longwave sound. The stationary flows occurring in this case may have a significant influence on the heat- and mass-transfer processes. In the present article we investigate the velocity field in the interior and exterior of a drop executing forced oscillatory motion as a result of its interaction with the ambient liquid. At a sufficiently large distance from the drop the ambient liquid oscillates in a specified way, where $s/R \ll 1$ (s is the amplitude of displacement of the liquid particles, and R is the radius of the drop).

The interface of the two media executes a complex motion consisting of its displacement as a whole and of deformation, i.e., of a departure of its shape from the initial spherical shape. Both liquids (inside and outside the drop) are assumed viscous and incompressible. There are no gravity forces. The flow pattern is assumed to be axisymmetric with respect to the straight line passing through the center of gravity of the drop and oriented along the direction of motion of the unperturbed liquid (in the spherical coordinate system used below the polar axis will coincide with the axis of symmetry). The motion of the liquid is assumed to be periodic in time.

The region under consideration is divided into two parts, the exterior (region outside the drop) and the interior (region inside the drop). All the quantities referring to the interior region except the independent variables are denoted by primes. The coordinate origin is fixed at the center of gravity of the drop. The initial equations for the exterior region are written in the form

$$\frac{\partial \mathbf{w}}{\partial t} + [(\mathbf{w} - \mathbf{v}_0) \nabla] \mathbf{w} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{w},$$
$$\nabla \mathbf{w} = 0,$$

where **w** is the velocity of the liquid particles in the fixed coordinate system, p is the pressure, ρ is the density, ν is the coefficient of kinematic viscosity, and $\mathbf{v}_0$ is the velocity of the center of gravity of the drop (this quantity must be determined during the solution of the problem).

The solution for the exterior region must be bounded; furthermore, at an infinitely large distance from the drop the condition

$$\mathbf{w} = \mathbf{u} \cos \omega t,$$

must be satisfied, where **u** is the amplitude of the velocity of the unperturbed liquid and ω is the angular frequency of the oscillations.

The initial equations for the interior region are similar in form. The solution for this region also must be bounded.

The following conditions must be satisfied at the interface of the two regions:

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 109-115, May-June, 1975. Original article submitted October 5, 1974.

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$$\mathbf{w}=\mathbf{w}';$$
$$n_k\Pi_{ik}-n_k\Pi'_{ik}=\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)n_i,$$

where Π_{ik} are the components of the stress tensor; n_i are the components of the unit vector normal to the surface of separation; α is the coefficient of surface tension; R_1, R_2 are the principal radii of curvature of the surface of separation; k, i = 1, 2, 3; repeated indices imply summation (in the second condition, see [1]).

In accordance with the methods of the perturbation theory we shall seek the solution of this problem in the form of power series of the small parameter s/R and we shall restrict ourselves to the first two terms of the expansion, i.e., we write w, w', p, p' in the form

$$\mathbf{w}=\mathbf{v}_1+\mathbf{v}_2;\ p=p_1+p_2;$$
$$\mathbf{w}'=\mathbf{v}'_1+\mathbf{v}'_2;\quad p'=p'_1+p'_2.$$

For the first approximation we have

$$\frac{\partial\mathbf{v}_1}{\partial t}=-\frac{1}{\rho}\nabla p_1+\nu\Delta\mathbf{v}_1,$$
$$\Delta\mathbf{v}_1=0.$$

In this approximation the boundary conditions are linear and homogeneous with respect to the components of the velocity vector. This follows from the fact that, since the pressure p_1 is linear in the components of $\mathbf{v}_1$ the components of the stress tensor Π_{ik} are also linear in $\mathbf{v}_1$ (the same holds for Π'_{ik}). It is well known [1] that in the first approximation

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{2}{R}-\frac{2\zeta}{R^2}-\frac{1}{R^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\zeta}{\partial\theta}\right),$$

where R is the radius of the unperturbed drop, ζ is the amount of deflection of a point on the surface of the drop from its mean (unperturbed position), and θ is the polar angle in the spherical coordinate system. Since

$$\zeta=\int v_{r1}dt,$$

where v_{r1} is the radial component of $\mathbf{v}_1$ at the surface of the drop, the expression for $(1/R_1)+(1/R_2)$ is linear in v_{r1}. The homogeneity of the boundary conditions follows from the fact that the pressure p_1 occurring in the expression for the components of the stress tensor is determined with an accuracy up to an arbitrary function of time.

In the first approximation the boundary conditions at the surface of the drop, whose form differs only slightly from spherical, can be taken to be the same as at the unperturbed sphere.

It may be stated that the solution in this approximation does not contain stationary components. Actually, by virtue of the linearity of the equations and the boundary conditions the problems of determining the stationary and oscillatory components are solved separately. The stationary component must vanish at a finite distance from the drop (by stipulation); therefore because of the homogeneity of the stationary problem the solution for these stationary components can only be zero.

Let us put $\mathbf{v}_1=\mathbf{v}\exp(-i\omega t)$; $p_1=p\exp(-i\omega t)$ (only the real part is meaningful), where $\mathbf{v}$ and p are functions of the coordinates (the same holds for $\mathbf{v}'_1$, p'_1). The equations for $\mathbf{v}$ and p are of the form

$$-i\omega\mathbf{v}+\frac{1}{\rho}\nabla p=\nu\Delta\mathbf{v},\qquad(1)$$
$$\nabla\mathbf{v}=0.$$

Similar equations are obtained also for $\mathbf{v}'$, p'. Taking account of the statement made above, the boundary conditions are written in the form

$$\mathbf{v}/_{r\to\infty}\to\mathbf{u};\ \mathbf{v}/_{r=R}=\mathbf{v}'/_{r=R};\qquad(2)$$

$$\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\bigg|_{r=R}=\mu'\left(\frac{1}{r}\frac{\partial v_r'}{\partial\theta}+\frac{\partial v_\theta'}{\partial r}-\frac{v_\theta'}{r}\right)\bigg|_{r=R};$$

$$-p+2\mu\frac{\partial v_r}{\partial r}\bigg|_{r=R}=-p'+2\mu'\frac{\partial v_r'}{\partial r}+\frac{\alpha}{R^2 i\omega}\left[2v_r+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right]\bigg|_{r=R}.$$

The formulation of the boundary conditions in form (2) (with the addition of the requirement of boundedness of the solution in both regions) permits complete solution of the problem in the first approximation; in particular, it is possible to determine the velocity of the drop as a whole and also the change of its form.

In the second approximation we shall consider only the stationary solution, since the oscillating solution is determined mainly by the first approximation [with an accuracy up to small quantities of the order $(s/R)^2$]. For the stationary flow we have the equations

$$\langle[(\mathbf{v}_1-\mathbf{v}_0)\nabla]\mathbf{v}_1\rangle=-\frac{1}{\rho}\nabla p_2+\nu\Delta\mathbf{v}_2, \qquad (3)$$
$$\nabla\mathbf{v}_2=0,$$

where the angular brackets denote averaging over the oscillation period. The equations for $\mathbf{v}_2'$, p_2' are also written in the same way.

Let us formulate the boundary conditions. The velocity $\mathbf{v}_2$ must vanish at an infinite distance from the drop. It is clear from the symmetry of the problem that the drop as a whole cannot execute stationary motion. Averaged over time the surface of the drop has a constant form only slightly differing from spherical; therefore it can be assumed (with an accuracy up to small quantities defined by the subsequent approximations) that averaged over time the drop is simply a sphere at rest in the outer liquid; then at the surface of the drop it is sufficient to require that the radial components of the velocity v_{2r} and v_{2r}' be zero and also the tangential components $v_{2\theta}$ and $v_{2\theta}'$ from and into the drop be equal.

The complete boundary conditions are then written in the form

$$\mathbf{v}_2|_{r\to\infty}\to 0;\quad v_{2r}|_{r=R}=v_{2r}'|_{r=R}=0;\quad v_{2\theta}|_{r=R}=v_{2\theta}'|_{r=R};$$
$$\mu\left(\frac{1}{r}\frac{\partial v_{2r}}{\partial\theta}+\frac{\partial v_{2\theta}}{\partial r}-\frac{v_{2\theta}}{r}\right)\bigg|_{r=R}=\mu'\left(\frac{1}{r}\frac{\partial v_{2r}'}{\partial\theta}+\frac{\partial v_{2\theta}'}{\partial r}-\frac{v_{2\theta}'}{r}\right)\bigg|_{r=R}.$$

These relations must be supplemented by the requirement that the solution be bounded in both regions.

We now turn to the solution of the problem in the first approximation. We shall show that in this approximation the drop will retain the spherical shape. If it is actually so, then the solution satisfying Eqs. (1) and boundary conditions (2) can be written in the form

$$v_r=f(r)\cos\theta;\quad v_\theta=\varphi(r)\sin\theta;\quad p=\psi(r)\cos\theta.$$

The factor in the surface-tension coefficient α in (2) vanishes identically, i.e., the solution is independent of α. That this result is not in contradiction with the assumption of sphericity of the drop (under the condition that the solution is unique) proves the validity of that assumption.

Batchelor [2] has pointed out a similar fact for the case of stationary motion of the drop in a viscous liquid at small Reynolds numbers (Hadamard–Rybchinskii problem). In both cases this is a result of neglecting the nonlinear terms in the problem.

The solution for the oscillatory components, which is bounded and satisfies the boundary conditions at infinity, is of the form

$$v_r=u\cos\theta\left[a\exp(ikr)\left(\frac{2}{ikr^3}-\frac{2}{r^2}\right)-\frac{2b}{r^3}+1\right]; \qquad (4)$$
$$v_\theta=u\sin\theta\left[a\exp(ikr)\left(\frac{1}{ikr^3}-\frac{1}{r^2}+\frac{ik}{r}\right)-\frac{b}{r^3}-1\right];$$
$$p=u\cos\theta\mu k^2\left(\frac{b}{r^2}+r\right);$$

$$v_r' = 2u\cos\theta\left\{c\left[\exp(i\varkappa r)\left(\frac{1}{r^3}-\frac{i\varkappa}{r^2}\right)-\exp(-i\varkappa r)\left(\frac{1}{r^3}+\frac{i\varkappa}{r^2}\right)\right]-d\right\};$$

$$v_\theta' = u\sin\theta\left\{c\left[\exp(i\varkappa r)\left(\frac{1}{r^3}-\frac{i\varkappa}{r^2}-\frac{\varkappa^2}{r}\right)-\exp(-i\varkappa r)\left(\frac{1}{r^3}+\frac{i\varkappa}{r^2}-\frac{\varkappa^2}{r}\right)\right]+2d\right\};$$

$$p' = -2du\cos\theta\mu'\varkappa^2 r.$$

Here $k = (1+i)/\delta$; $\varkappa = (1+i)/\delta'$, $\delta = \sqrt{\nu/2\omega}$; a, b, c, d are undetermined constants which must be found from the boundary conditions at the surface of the drop (2). Below we shall assume that $|kR| \gg 1$, $|\varkappa R| \gg 1$; then we have

$$a = -\frac{3iR}{k}\exp(ikR)\frac{\left(1-\frac{\rho'}{\rho}\right)\left(2\sigma-\frac{\mu'}{\mu}\right)}{\left(1+2\frac{\rho'}{\rho}\right)\left(\frac{\delta}{\varepsilon}+\frac{\mu'}{\mu}\right)};\tag{5}$$

$$c = \frac{3}{\varepsilon i\varkappa^3}\exp(i\varkappa R)\frac{\left(1-\frac{\rho'}{\rho}\right)}{\left(1+2\frac{\rho'}{\rho}\right)\left(\frac{\sigma}{\varepsilon}+\frac{\mu'}{\mu}\right)};$$

$$b = -\frac{R^3\left(1-\frac{\rho'}{\rho}\right)}{1+2\frac{\rho'}{\rho}};$$

$$d = -\frac{3}{2\left(1+2\frac{\rho'}{\rho}\right)},\ \text{where}\ \varepsilon = \frac{1}{ikR};\quad \sigma = \frac{1}{i\varkappa R}.$$

The velocity of The drop v_0 is determined from the first or fourth formula in (4) for r = R and is given by

$$\mathbf{v}_0 = \frac{3\rho}{\rho+2\rho'}\mathrm{u}\cos\omega t,$$

which coincides with the well-known formula for the velocity of a solid sphere of density ρ' located in an ideal oscillating liquid of density ρ.

In order to find the stationary components of the flow $\mathbf{v}_2$, $\mathbf{v}_2'$ we rewrite Eqs. (3) in the form

$$\nu\Delta\,\mathrm{rot}\,\mathbf{v}_2 = -\langle\mathrm{rot}\,[(\mathbf{v}_1-\mathbf{v}_0)\times\mathrm{rot}\,\mathbf{v}_1]\rangle = \varepsilon\varphi(r)\sin 2\theta,\tag{6}$$

where ε is the azimuthal base vector in the spherical coordinate system. For $\varphi(r)$ and $\varphi'(r)$ we then get

$$\varphi(r) = \frac{u^2}{\delta^2 R^3}\exp(-\eta)\left\{\frac{a_1^2+a_2^2}{\delta}[\cos\eta-\sin\eta-\exp(-\eta)] + \frac{3b\eta}{R^2}[a_1(\cos\eta-\sin\eta)-a_2(\cos\eta+\sin\eta)]\right\};$$

$$\varphi_r' = \frac{2u^2}{\delta'^5 R^3}\exp(\eta')\left(c_1^2+c_2^2\right)[\exp(\eta')-\sin\eta'-\cos\eta'].$$

Here $\eta = (r-R)/\delta$, $\eta' = (r-R)/\delta'$; the real constants a_1, a_2, c_1, c_2 are related to constants a, c in (5) in the following manner:

$$a_1+ia_2 = a\exp(ikR);\quad c_1+ic_2 = c\exp(-i\varkappa R).$$

Introducing the stream function ψ [$v_{2r} = (1/r^2\sin\theta)(\partial\psi/\partial\theta)$, $v_{2\theta} = (1/r\sin\theta)(\partial\psi/\partial r)$] and expressing it in the form

$$\psi = \Phi(r)\sin 2\theta\sin\theta,$$

we rewrite Eq. (6) in the form

$$\frac{\Phi_{rrrr}}{r} - 12\frac{\Phi_{rr}}{r^3} + 24\frac{\Phi_r}{r^4} = \frac{\varphi(r)}{\nu}.\tag{7}$$

The boundary conditions at the surface of the drop become

$$\Phi|_{r=R} = \Phi'|_{r=R} = 0; \tag{8}$$
$$\Phi_r|_{r=R} = \Phi'_r|_{r=R};$$
$$\mu\left(\frac{\Phi_{rr}}{R} - 2\frac{\Phi_r}{R^2}\right)\Big|_{r=R} = \mu'\left(\frac{\Phi'_{rr}}{R} - 2\frac{\Phi'_r}{R^2}\right)\Big|_{r=R}.$$

The solution for Φ and Φ' that is bounded and satisfies the boundary condition at infinity ($\mathbf{v}_2 \to D$) is written in the form

$$\Phi = A/r^2 + B + \Phi^0; \tag{9}$$
$$\Phi' = Cr^5 + Dr^3 + \Phi'^0,$$

where Φ^0 and Φ'^0 are particular solutions of (7) given by the formulas

$$\Phi^0 = \frac{R}{\nu}\iiiint \varphi(r)\,dr;$$
$$\Phi'^0 = \frac{R}{\nu'}\iiiint \varphi'(r)\,dr.$$

Obviously, the particular solutions thus obtained satisfy (under the condition $|kR| \gg 1$, $|\varkappa R| \gg 1$) Eqs. (7); A, B, C, D are undetermined constants that are found from boundary conditions (8). After some computations we get

$$\Phi^0 = \frac{u^2\delta^2}{\nu R^2}\left\{-\frac{a_1^2 + a_2^2}{4\delta}\left[\frac{1}{4}\exp(-2\eta) + \exp(-\eta)(\cos\eta - \sin\eta)\right] + \right.$$
$$\left. + \frac{3b}{R^2}\left[\frac{a_2 - a_1}{4}\exp(-\eta)\,\eta\cos\eta + \frac{a_1 + a_2}{4}\exp(-\eta)\,\eta\sin\eta + a_1\exp(-\eta)\sin\eta + a_2\exp(-\eta)\cos\eta\right]\right\};$$
$$\Phi'^0 = \frac{u^2(c_1^2 + c_2^2)}{\nu'\delta' R^2}\left[\frac{1}{8}\exp(2\eta') + \frac{1}{2}\exp(\eta')(\cos\eta' + \sin\eta')\right];$$
$$A = \frac{R^3\left(\alpha_2 R^2 - 5\frac{\mu'}{\mu}\alpha_1\right)}{10\left(1 + \frac{\mu'}{\mu}\right)}; \quad B = -\frac{R^3\left(\alpha_2 - 5\frac{\mu'}{\mu}\frac{\alpha_1}{R^2}\right)}{10\left(1 + \frac{\mu'}{\mu}\right)};$$
$$C = -\frac{\alpha_2 + 5\frac{\alpha_1}{R^2}}{10R^2\left(1 + \frac{\mu'}{\mu}\right)}; \quad D = \frac{\alpha_2 R^2 + 5\alpha_1}{10R^2\left(1 + \frac{\mu'}{\mu}\right)},$$

where

$$\alpha_1 = \frac{u^2}{4R^2}\left[5\frac{c_1^2 + c_2^2}{\nu'\delta'^2} - \frac{5}{2\nu}(a_1^2 + a_2^2) - 9\frac{b\delta}{R^2\nu}(a_1 - a_2)\right];$$
$$\alpha_2 = \frac{u^2}{\nu R^3}\left[\frac{3}{4\delta}(a_1^2 + a_2^2) + \frac{3ba_1}{R^2}\right] + \frac{3}{2}\frac{\mu'}{\mu}\frac{u^2(c_1^2 + c_2^2)}{\nu'\delta'^3 R^3}.$$

In particular, if $\mu'/\mu \to \infty$, $\rho'/\rho \to \infty$ (solid sphere at rest in an oscillating liquid), then according to (5) we have

$$a_1 = a_2 = -\frac{3}{4}R\delta; \quad b = \frac{R^3}{2}; \quad c_1 = 0; \quad c_2 = \frac{3\delta'^3 R}{4\delta\frac{\mu'}{\mu}}.$$

Then

$$A = \frac{45}{8\cdot 16}\frac{u^2\delta^2 R^3}{\nu}; \quad B = -\frac{45u^2\delta^2 R}{8\cdot 16\nu}; \quad C = 0; \quad D = 0.$$

Inside the drop there is no flow. Let us determine the tangential component of the velocity $v_{2\theta}$ at the limit of the "boundary layer" ($\eta \gg 1$) in the outer liquid. With the use of (9) we get

$$v_{2\theta}(\delta) = \frac{1}{R}\Phi_r\sin 2\theta = -\frac{2A}{R^4}\sin 2\theta = -\frac{45}{32}\frac{u^2}{R\omega}\sin 2\theta,$$

which coincides with the result obtained in [3]. If $\mu'/\mu \to 0$, $\rho'/\rho \to 0$ (model of a bubble in the liquid), then we have

$$a_1=0;\ a_2=3\delta^2;\ c_1=0;\ c_2=-3/2R\delta^2;\ b=-R^3$$

and

$$C=-\frac{45u^2\delta'^2}{32\nu' R^4};\quad D=\frac{45u^2\delta'^2}{32\nu' R^2}.$$

In order of magnitude the constants A and B are equal to

$$A\sim\frac{u^2\delta^3R^2}{\nu};\quad B\sim\frac{u^2\delta^3}{\nu}.$$

For $\eta' \ll -1$ the tangential component of the velocity inside the drop is equal to

$$v'_{2\theta}(\delta')=-\frac{45}{8}\frac{u^2}{R\omega}\sin 2\theta.$$

Outside the drop the velocity $v_{2\theta}(\delta)$ has the following order of magnitude:

$$v_{2\theta}(\delta)\sim\frac{u^2}{R\omega}\frac{\delta}{R},$$

hence

$$|v_{2\theta}(\delta)|\ll|v'_{2\theta}(\delta')|.$$

The author expresses gratitude to V. E. Nakoryakov for constant attention and assistance in the work.

LITERATURE CITED

1. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], GITTL, Moscow (1953), pp. 119,287,292.
2. G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press (1970).
3. V. E. Nakoryakov, A. P. Burdukov, A. M. Boldarev, and P. N. Terleev, Heat and Mass Transfer in an Acoustic Field [in Russian], Inst. Teplofiz., Sibirsk. Otd., Akad. Nauk SSSR, Novosibirsk (1970), p. 18.